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Generalized forms and vector fields

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Abstract
The generalized vector is defined on an n-dimensional manifold. The interior product and Lie derivative acting on generalized p-forms, $-1 \leqslant p \leqslant n$ are introduced. The generalized commutator of two generalized vectors is defined. Adding a correction term to Cartan's formula, the generalized Lie derivative's action on a generalized vector field is defined. We explore various identities of the generalized Lie derivative with respect to generalized vector fields, and discuss an application.

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1. Introduction

The idea of a -1-form, i.e. a form of negative degree, was first introduced by Sparling [1, 2] during an attempt to associate an abstract twistor space with any real analytic spacetime obeying vacuum Einstein's equations. Sparling obtained these forms from the equation of the twistor surfaces without torsion. There he assumed the existence of a -1-form.

Nurowski and Robinson [3, 4] took this idea and used it to develop a structure of generalized differential forms. They studied Cartan's structure equation, Hodge star operator, codifferential and Laplacian operators on the space of generalized differential forms. They found a number of physical applications to mechanical and physical field theories such as the generalized form of Hamiltonian systems, scalar fields, Maxwell and Yang–Mills fields and Einstein's vacuum field equations. Guo *et al* [5] later found that Chern–Simons theories can be related to gravity using the language of generalized forms. Robinson [6] extended the algebra and calculus of generalized differential forms to type N, where N is the number of independent -1-form fields in the structure. $N = 1$ represents the ordered pair representation of [3, 4]. We also work in the $N = 1$ case. The action of ordinary vector fields on generalized forms is discussed by Nurowski and Robinson [4]—i.e. the interior product and Lie derivative of generalized forms with respect to ordinary vector fields. In this paper we introduce the generalized vector field V as an ordered pair of an ordinary vector field v_1 and ordinary scalar field v_0, $V = (v_1, v_0)$.

In section 2 we briefly discuss generalized forms, their products and exterior derivative developed by Sparling and Nurowski–Robinson. In section 3 we define the interior product of a generalized p-form with a generalized vector field as a mapping from generalized p-forms to generalized $p - 1$-forms and various identities are discussed. In section 4 we define the generalized Lie derivative of a generalized p-form using Cartan's formula. It follows that a generalized bracket of two generalized vector fields can be defined and generalized vector fields form a Lie algebra with this generalized bracket. But the Lie derivative of a vector field cannot be defined using Cartan's formula. So in section 5 we add a correction term with Cartan's formula and define a new and improved generalized Lie derivative. This allows us to define the Lie derivative of a generalized vector field. The Lie derivative satisfies the Leibniz rule and linearity, while the definition of generalized commutator remains the same. In section 6 we give an alternative construction of the generalized vector field assuming the existence of a special kind of vector field. Its properties are discussed and the results compared with the relations obtained earlier. Finally, an example is given in section 7, where we discuss the Hamiltonian of the relativistic free particle in the language of generalized forms and generalized vector fields.

2. Generalized *p*-forms

Let M be a smooth n-dimensional manifold, $C^\infty(M)$ the ring of real-valued smooth functions, $\mathcal{X}(M)$ the Lie algebra of vector fields and $\Omega^p(M)$ the $C^\infty(M)$-module of differential p-forms, $1 \leqslant p \leqslant n$.

A generalized p-form $\overset{p}{\mathfrak{a}}$ on M is defined [3, 4] as an ordered pair of an ordinary p-form α_p and an ordinary $p + 1$-form α_{p+1},

$$\overset{p}{\mathfrak{a}} = (\alpha_p, \alpha_{p+1}) \qquad -1 \leqslant p \leqslant n. \tag{1}$$

Since there is no such thing as an ordinary -1-form or an ordinary $n + 1$-form (Sparling's -1-form is by no means ordinary, as we shall discuss later),

$$\overset{-1}{\mathfrak{a}} = (0, \alpha_0), \qquad \overset{n}{\mathfrak{a}} = (\alpha_n, 0). \tag{2}$$

We will denote the space of generalized p-form fields on M by $\Omega^p_G(M)$.

The wedge product of a generalized p-form $\overset{p}{\mathfrak{a}} = (\alpha_p, \alpha_{p+1})$ and a generalized q-form $\overset{q}{\mathfrak{b}} = (\beta_q, \beta_{q+1})$ is a map $\wedge : \Omega^p_G(M) \times \Omega^q_G(M) \to \Omega^{p+q}_G(M)$ defined as [3]

$$\overset{p}{\mathfrak{a}} \wedge \overset{q}{\mathfrak{b}} = (\alpha_p\beta_q, \alpha_p\beta_{q+1} + (-1)^q\alpha_{p+1}\beta_q). \tag{3}$$

Here and below, we write $\alpha_p\beta_q$ for the (ordinary) wedge product of ordinary forms α_p and β_q, without the $\wedge$ symbol, so as to keep the equations relatively uncluttered. We will use the $\wedge$ symbol only to denote the wedge product of generalized forms. Clearly, the wedge product as defined above satisfies

$$\overset{p}{\mathfrak{a}} \wedge \overset{q}{\mathfrak{b}} = (-1)^{pq}\overset{q}{\mathfrak{b}} \wedge \overset{p}{\mathfrak{a}}, \tag{4}$$

exactly like the wedge product of ordinary forms.

The generalized exterior derivative $\mathbf{d} : \Omega^p_G \to \Omega^{p+1}_G$ is defined as

$$\mathbf{d}\overset{p}{\mathfrak{a}} = (\mathrm{d}\alpha_p + (-1)^{p+1}k\alpha_{p+1}, \mathrm{d}\alpha_{p+1}). \tag{5}$$

Here d is the ordinary exterior derivative acting on ordinary p-forms and k is a non-vanishing constant. It is easy to check that the generalized exterior derivative satisfies the Leibniz rule on generalized forms:

$$\mathbf{d}(\overset{p}{\mathfrak{a}} \wedge \overset{q}{\mathfrak{b}}) = (\mathbf{d}\overset{p}{\mathfrak{a}}) \wedge \overset{q}{\mathfrak{b}} + (-1)^p \overset{p}{\mathfrak{a}} \wedge (\mathbf{d}\overset{q}{\mathfrak{b}}). \tag{6}$$

We make a small digression here to note that there is an alternative definition of the generalized p-forms we mentioned at the beginning. In this we start by assuming, following Sparling [1], the existence of a (-1)-form field, i.e. a form of degree -1. Denoting this form by ζ and using $\mathrm{d}^2 = 0$, we find that we should have $\mathrm{d}\zeta = k$, with k being a constant. Then a generalized p-form is defined in terms of an ordinary p-form α_p and an ordinary $(p+1)$-form α_{p+1} as

$$\overset{p}{\mathfrak{a}} = \alpha_p + \alpha_{p+1} \wedge \zeta. \tag{7}$$

Note that ζ is not a differential form in the usual sense, i.e. given an ordinary one-form α_1, the wedge product $\alpha_1 \wedge \zeta$ is not a function on the manifold. However, if we nevertheless treat the object of equation (7) as an ordinary p-form, we find using $\zeta \wedge \zeta = 0$ that the wedge product of two such forms is

$$\overset{p}{\mathfrak{a}} \wedge \overset{q}{\mathfrak{b}} = \alpha_p \beta_q + (\alpha_p \beta_{q+1} + (-1)^q \alpha_{p+1} \beta_q) \wedge \zeta, \tag{8}$$

while the exterior derivative of such a p-form works out to be

$$\mathbf{d}\overset{p}{\mathfrak{a}} = \mathrm{d}\alpha_p + (-1)^{p+1} k \alpha_{p+1} + \mathrm{d}\alpha_{p+1} \wedge \zeta. \tag{9}$$

The two definitions in equations (1) and (7) are clearly equivalent. We will employ the first definition in all calculations.

3. Generalized vectors and contraction

In order to go beyond forms and define generalized tensor fields, we will need to define a generalized vector field. We could define a generalized vector as an ordered pair of a vector and a bivector, dual to a generalized one-form, referring either to the ring of real functions, or to the ring of generalized zero-forms. Both these definitions quickly run into problems. The interior product with generalized forms is ill-defined in one case, and fails to satisfy Leibniz rule in the other. In this paper we propose a new definition of the generalized vector field.

3.1. Generalized vector fields

We define a generalized vector field V to be an ordered pair of an ordinary vector field v_1 and an ordinary scalar field v_0:

$$V := (v_1, v_0), \qquad v_1 \in \mathcal{X}(M), \quad v_0 \in C^\infty(M). \tag{10}$$

Clearly, the submodule $v_0 = 0$ of generalized vector fields can be identified with the module of ordinary vector fields on the manifold. We will write $\mathcal{X}_G(M)$ for the space of generalized vector fields on M. On this space, ordinary scalar multiplication is defined by

$$\lambda V = (\lambda v_1, \lambda v_0), \tag{11}$$

as expected. On the other hand, we can define *generalized scalar multiplication* by a generalized zero-form $\overset{0}{\mathfrak{a}} = (\alpha_0, \alpha_1)$, by

$$\overset{0}{\mathfrak{a}} V = (\alpha_0 v_1, \alpha_0 v_0 + i_{v_1} \alpha_1) \in \mathcal{X}_G(M), \tag{12}$$

where i is the usual contraction operator. Note that on the submodule $v_0 = 0$,, i.e. on ordinary vector fields, the generalized scalar multiplication is different from the ordinary scalar multiplication given in equation (11). The generalized scalar multiplication is linear, and satisfies

$$\overset{0}{\mathfrak{a}}(\overset{0}{\mathfrak{b}}V) = (\overset{0}{\mathfrak{a}} \wedge \overset{0}{\mathfrak{b}})V. \tag{13}$$

3.2. Generalized contraction

Next we define the generalized contraction, or interior product I_V, with respect to a generalized vector $V = (v_1, v_0)$, as a map from generalized p-forms to generalized $(p-1)$-forms. We will define this mapping in such a way that, given a generalized p-form $\overset{p}{\mathfrak{a}} = (\alpha_p, \alpha_{p+1})$, setting $v_0 = 0$ and $\alpha_{p+1} = 0$ gives the contraction formula of an ordinary vector with an ordinary p-form. Then we can define the most general contraction formula $I_V : \Omega_G^p \to \Omega_G^{p-1}$ as

$$I_V \overset{p}{\mathfrak{a}} = (i_{v_1}\alpha_p, \lambda(p) i_{v_1}\alpha_{p+1} + \tau(p) v_0 \alpha_p), \tag{14}$$

where $\lambda(p)$ and $\tau(p)$ are unknown functions.

Now we impose the Leibniz rule on the contraction formula, so that I_V is a $\wedge$-anti-derivation:

$$I_V(\overset{p}{\mathfrak{a}} \wedge \overset{q}{\mathfrak{b}}) = (I_V \overset{p}{\mathfrak{a}}) \wedge \overset{q}{\mathfrak{b}} + (-1)^p \overset{p}{\mathfrak{a}} \wedge (I_V \overset{q}{\mathfrak{b}}). \tag{15}$$

We find from this condition, using equations (3) and (14), that

$$\lambda(p) = 1, \tag{16}$$

and

$$(-1)^q \tau(p) + (-1)^p \tau(q) = \tau(p+q). \tag{17}$$

The non-trivial solution of this is $\tau(p) = p(-1)^{p-1}$, up to an arbitrary constant, which we will set to unity.

Thus the formula for contraction of a generalized vector field with a generalized p-form is

$$I_V \overset{p}{\mathfrak{a}} = \big(i_{v_1}\alpha_p, i_{v_1}\alpha_{p+1} + p(-1)^{p-1} v_0 \alpha_p\big). \tag{18}$$

The interior product is linear under ordinary scalar multiplication,

$$I_{V+\mu W} = I_V + \mu I_W, \tag{19}$$

where μ is an ordinary scalar field, but it fails to satisfy linearity under generalized scalar multiplication,

$$\begin{aligned} I_{\overset{0}{\mathfrak{a}}V} \overset{q}{\mathfrak{b}} &\neq \overset{0}{\mathfrak{a}}(I_V \overset{q}{\mathfrak{b}}) \\ &\neq I_V(\overset{0}{\mathfrak{a}} \wedge \overset{q}{\mathfrak{b}}). \end{aligned} \tag{20}$$

The generalized interior product, when restricted to the ordinary subspace of vector fields, i.e. to generalized vector fields of the type $V = (v, 0)$, is identical to the interior product of ordinary differential geometry, acting independently on the two ordinary forms which make up a generalized form. However, several relations which hold for the ordinary interior product are not satisfied by the generalized one; we list a few here.

(i) The generalized interior product does not anticommute; for two generalized vector fields $V = (v_1, v_0)$ and $W = (w_1, w_0)$ and any generalized p-form $\overset{p}{\mathfrak{a}} = (\alpha_p, \alpha_{p+1})$,

$$(I_V I_W + I_W I_V)\overset{p}{\mathfrak{a}} = (-1)^{p-1}\big(w_0 i_{v_1} + v_0 i_{w_1}\big)(0, \alpha_p) \neq 0. \tag{21}$$

(ii) The contraction of the generalized zero-form does not vanish:

$$I_V \overset{0}{\mathfrak{a}} = \big(0, i_{v_1}\alpha_1\big). \tag{22}$$

(iii) The contraction of a generalized one-form is a generalized scalar:

$$I_V \overset{1}{\mathfrak{a}} = \big(i_{v_1}\alpha_1, v_0\alpha_1 + i_{v_1}\alpha_2\big). \tag{23}$$

It follows that the space of generalized vector fields $\mathcal{X}_G(M)$ is not the dual space of generalized 1-form fields $\Omega^1_G(M)$.

4. Lie derivative: first attempt

Equipped with the generalized exterior derivative and the generalized inner derivative we can define a generalization of the Lie derivative. In his book [7], Cartan introduces a combination of the exterior derivative and the interior product which was coined as the 'Lie derivative' by Sledbodzinsky. Marsden calls this Cartan's 'magic formula'. The Lie derivative with respect to a vector field v acts on exterior differential forms according to Cartan's magic formula:

$$L_v\alpha = i_v\,\mathrm{d}\alpha + \mathrm{d}(i_v\alpha). \tag{24}$$

This formula does not depend on the metric imposed upon the base space of independent variables, and has been called the homotopy formula by Arnold.

We will take as our starting point Cartan's formula for the Lie derivative of a p-form with respect to a vector, and generalize that. However, in section 5 we will find that the resulting derivative is problematic when applied on a generalized vector field and we have to add an extra correction term.

For the moment, let us define the generalized Lie derivative $\mathcal{L}_V$ with respect to the generalized vector field V, $\mathcal{L}_V : \Omega^p_G(M) \to \Omega^p_G(M)$, as

$$\mathcal{L}_V \overset{p}{\mathfrak{a}} = I_V \mathbf{d}\overset{p}{\mathfrak{a}} + \mathbf{d} I_V \overset{p}{\mathfrak{a}}, \tag{25}$$

which is a generalization of Cartan's formula. Using equations (5) and (18), we find that

$$\begin{aligned}\mathcal{L}_V \overset{p}{\mathfrak{a}} = \big(&L_{v_1}\alpha_p - pkv_0\alpha_p, L_{v_1}\alpha_{p+1} - (p+1)kv_0\alpha_{p+1} \\ &+ p(-1)^{p-1}(\mathrm{d}v_0)\alpha_p + (-1)^p v_0\,\mathrm{d}\alpha_p\big),\end{aligned} \tag{26}$$

where as usual $\overset{p}{\mathfrak{a}} = (\alpha_p, \alpha_{p+1})$, $V = (v_1, v_0)$, and L_{v_1} is the ordinary Lie derivative with respect to the ordinary vector field v_1. This formula will be modified later on, but let us check the consequences of this formula here.

This generalized Lie derivative is a derivation on the space of generalized forms, satisfying the Leibniz rule and linearity:

$$\mathcal{L}_V(\overset{p}{\mathfrak{a}} \wedge \overset{q}{\mathfrak{b}}) = (\mathcal{L}_V\overset{p}{\mathfrak{a}}) \wedge \overset{q}{\mathfrak{b}} + \overset{p}{\mathfrak{a}} \wedge (\mathcal{L}_V\overset{q}{\mathfrak{b}}), \tag{27}$$

$$\mathcal{L}_{\lambda V+W} = \lambda\mathcal{L}_V + \mathcal{L}_W, \tag{28}$$

where V, W are generalized vector fields and λ is an arbitrary constant.

4.1. Generalized commutator and Jacobi identity

We recall that the ordinary Lie derivatives L_{v_1} and L_{w_1} with respect to ordinary vector fields v_1 and w_1 acting on an ordinary p-form α_p satisfy

$$L_{v_1} L_{w_1} \alpha_p - L_{w_1} L_{v_1} \alpha_p = L_{[v_1, w_1]} \alpha_p, \tag{29}$$

where $[v_1, w_1]$ is the usual commutator between the ordinary vector fields v_1 and w_1.

We will generalize this formula to compute the generalized commutator of two generalized vector fields. We will compute the generalized bracket

$$[\mathcal{L}_V, \mathcal{L}_W] \overset{p}{\mathfrak{a}} := (\mathcal{L}_V \mathcal{L}_W - \mathcal{L}_W \mathcal{L}_V) \overset{p}{\mathfrak{a}}, \tag{30}$$

and see if we can find a vector field $\{V, W\}$ such that $[\mathcal{L}_V, \mathcal{L}_W] = \mathcal{L}_{\{V,W\}}$. We will then define $\{V, W\}$ as the generalized commutator of generalized vector fields $V, W \in \mathcal{X}_G(M)$.

Expanding the right-hand side of equation (30), we find that it does have the form of a generalized Lie derivative of $\overset{p}{\mathfrak{a}}$ with respect to a generalized vector field. So we can, in fact, write

$$\mathcal{L}_V \mathcal{L}_W \overset{p}{\mathfrak{a}} - \mathcal{L}_W \mathcal{L}_V \overset{p}{\mathfrak{a}} = \mathcal{L}_{\{V,W\}} \overset{p}{\mathfrak{a}}, \tag{31}$$

where $\{V, W\}$ is a generalized vector field, the generalized commutator of V and W. For $V = (v_1, v_0)$, $W = (w_1, w_0)$, the generalized commutator is calculated directly using equation (26) to be

$$\{V, W\} = \big([v_1, w_1], L_{v_1} w_0 - L_{w_1} v_0\big). \tag{32}$$

We get back the ordinary commutation relation for ordinary vector fields by setting $v_0, w_0 = 0$. This generalized commutation relation can be identified with the Lie algebra of a semidirect product $\mathcal{X}(M) \ltimes C^\infty(M)$.

Generally speaking, let (v_1, w_1) be the elements of the Lie algebra of vector fields on M, and (v_0, w_0) the elements of $C^\infty(M)$, which is an Abelian Lie algebra under addition. The Lie algebra of the semidirect product is defined by equation (32).

We can check from its definition that $\{V, W\}$ is antisymmetric in V and W and bilinear. And we can also calculate directly that Jacobi identity is satisfied, for $U, V, W \in \mathcal{X}_G(M)$,

$$\{U, \{V, W\}\} + \{V, \{W, U\}\} + \{W, \{U, V\}\} = 0. \tag{33}$$

Therefore the space $\mathcal{X}_G(M)$ of generalized vector fields together with the generalized commutator $\{\,,\}$ forms a Lie algebra.

5. Lie derivative: improvement

While we could use Cartan's formula for a definition of the Lie derivative on forms, we have to find another definition for the Lie derivative of a vector field. We will do this by assuming that the generalized Lie derivative is a derivation on generalized vector fields.

In other words, we want that the following equality should hold for any two generalized vector fields V, W, and any generalized p-form $\overset{p}{\mathfrak{a}}$:

$$\mathcal{L}_V (I_W \overset{p}{\mathfrak{a}}) = I_W (\mathcal{L}_V \overset{p}{\mathfrak{a}}) + I_{\mathcal{L}_V W} \overset{p}{\mathfrak{a}}, \tag{34}$$

where we have written $\mathcal{L}_V W$ for the action of $\mathcal{L}_V$ on W. This is what we would like to define as the Lie derivative of W with respect to V. However, from equations (18) and (26) we find that

$$\mathcal{L}_V (I_W \overset{p}{\mathfrak{a}}) - I_W \mathcal{L}_V \overset{p}{\mathfrak{a}} = I_{([v_1, w_1] + k v_0 w_1, L_{v_1} w_0 - L_{w_1} v_0)} \overset{p}{\mathfrak{a}} - (-1)^p \big(0, L_{v_0 w_1} \alpha_p\big). \tag{35}$$

We see that $\mathcal{L}_V$ on generalized vectors cannot be defined if the Lie derivative of a generalized p-form is as in equation (26). We can try to resolve this problem by modifying the formula for the Lie derivative of a generalized p-form, by adding an extra term of the form $(0, \beta_{p+1})$ to the right-hand side of equation (26). This ordinary $p+1$-form β_{p+1} must be constructed form v_0 and α_p, such that setting $v_0 = 0$ implies $\beta_{p+1} = 0$. Also, this β_{p+1} should not depend on α_{p+1} or v_1 since the extra term in equation (35) does not depend on these objects. Then most general expression for β_{p+1} is

$$\beta_{p+1} = (-1)^p(\gamma(p)v_0\,\mathrm{d}\alpha_p + \delta(p)(\mathrm{d}v_0)\alpha_p). \tag{36}$$

With this extra term, we can define the modified Lie derivative $\hat{\mathcal{L}}_V$ as

$$\hat{\mathcal{L}}_V \overset{p}{\mathfrak{a}} = \mathcal{L}_V \overset{p}{\mathfrak{a}} + (0, \beta_{p+1}). \tag{37}$$

Now if we calculate $\hat{\mathcal{L}}_V I_W \overset{p}{\mathfrak{a}} - I_W \hat{\mathcal{L}}_V \overset{p}{\mathfrak{a}}$, we find that we must have $\delta(p) = p$ and $\gamma(p) = -1$ in order that $\hat{\mathcal{L}}_V W$ is well defined. Therefore we get

$$\begin{aligned}\hat{\mathcal{L}}_V \overset{p}{\mathfrak{a}} &= \mathcal{L}_V \overset{p}{\mathfrak{a}} + (-1)^p(0, -v_0\,\mathrm{d}\alpha_p + p\,\mathrm{d}v_0\,\alpha_p)\\ &= \big(L_{v_1}\alpha_p - pkv_0\alpha_p, L_{v_1}\alpha_{p+1} - (p+1)kv_0\alpha_{p+1}\big).\end{aligned} \tag{38}$$

This is a modification of the formula given in equation (26).

It can be checked easily using equation (38) that this new and improved generalized Lie derivative satisfies the Leibniz rule,

$$\hat{\mathcal{L}}_V(\overset{p}{\mathfrak{a}} \wedge \overset{q}{\mathfrak{b}}) = (\hat{\mathcal{L}}_V \overset{p}{\mathfrak{a}}) \wedge \overset{q}{\mathfrak{b}} + \overset{p}{\mathfrak{a}} \wedge (\hat{\mathcal{L}}_V \overset{q}{\mathfrak{b}}), \tag{39}$$

where $V \in \mathcal{X}_G$, $\overset{p}{\mathfrak{a}} \in \Omega_G^p$ and $\overset{q}{\mathfrak{b}} \in \Omega_G^q$.

We can now write the generalized Lie derivative of a generalized vector field. Using equations (18) and (38) we find

$$\hat{\mathcal{L}}_V I_W - I_W \hat{\mathcal{L}}_V = I_{([v_1,w_1]+kv_0w_1, L_{v_1}w_0)}. \tag{40}$$

Therefore we can define the generalized Lie derivative of a generalized vector field as

$$\hat{\mathcal{L}}_V W = \big([v_1, w_1] + kv_0w_1, L_{v_1}w_0\big), \tag{41}$$

where $V = (v_1, v_0)$, $W = (w_1, w_0)$, and $v_1, w_1 \in \mathcal{X}(M)$, $v_0, w_0 \in C^\infty(M)$.

Note that $v_0, w_0 = 0$ gives the ordinary Lie derivative of an ordinary vector field. Also, the new generalized Lie derivative on generalized vector fields is not the generalized commutator. It can be checked using the new and improved definition that the commutator of two generalized Lie derivatives is also a generalized Lie derivative as in equation (31) and the definition of the generalized commutator of two generalized vectors remains the same as in equation (32),

$$\hat{\mathcal{L}}_V \hat{\mathcal{L}}_W - \hat{\mathcal{L}}_W \hat{\mathcal{L}}_V = \hat{\mathcal{L}}_{\{V,W\}} \qquad V, W \in \mathcal{X}_G. \tag{42}$$

Note that both sides of this equation can be taken to act either on a generalized p-form or on a generalized vector.

6. Alternative definition of a generalized vector field

In section 2 we defined a generalized p-form as an ordered pair of an ordinary p-form and an ordinary $p+1$-form, and equipped the space of these objects with a generalized wedge product and a generalized exterior derivative. All calculations were done using this definition. As was mentioned there, we could have used an alternative definition, assuming the existence of a '-1- form' ζ such that $\mathrm{d}\zeta = k$, where k is a constant. Then a generalized p-form can be defined as $\overset{p}{\mathfrak{a}} = \alpha_p + \alpha_{p+1}\zeta$, and has identical properties with the generalized p-form defined as an ordered pair.

Coming back to generalized vector fields, we defined these objects as ordered pairs in section 3 and did all the calculations using this definition. The purpose of this section will be to give an alternative definition of the generalized vector field in analogy with the -1-form of Sparling *et al* and to show that the results found so far follow from using this definition also. Let us then define a generalized vector field as the sum of an ordinary vector field and a scalar multiple of a special vector field $\bar{X}$:

$$V = v_1 + v_0\bar{X}. \tag{43}$$

The vector field $\bar{X}$ is not an ordinary vector field, but may be thought of as the unit generalized vector. It will be defined by its action on ordinary p-forms,

$$i_{\bar{X}}\alpha_p = p(-1)^{p-1}\alpha_p \wedge \zeta, \tag{44}$$

for any $\alpha_p \in \Omega^p(M)$. Note that this formula for $i_{\bar{X}}$ automatically satisfies the Leibniz rule for contractions:

$$i_{\bar{X}}(\alpha_p\beta_q) = (i_{\bar{X}}\alpha_p)\beta_q + (-1)^p\alpha_p(i_{\bar{X}}\beta_q). \tag{45}$$

Note also that $i_{\bar{X}}\overset{p}{\mathfrak{a}} = i_{\bar{X}}\alpha_p$, where as usual, $\overset{p}{\mathfrak{a}} = \alpha_p + \alpha_{p+1}\zeta$.

We can define the contraction of a generalized p-form by an ordinary vector field $v_1 \in \mathcal{X}(M)$ as [4]

$$i_{v_1}\overset{p}{\mathfrak{a}} = i_{v_1}\alpha_p + \left(i_{v_1}\alpha_{p+1}\right) \wedge \zeta. \tag{46}$$

Using the definitions given in equations (44) and (46) we can now define the contraction of a generalized form by a generalized vector field $V = v_1 + v_0\bar{X}$ as

$$i_V\overset{p}{\mathfrak{a}} = i_{v_1}\alpha_p + \left(p(-1)^{p-1}v_0\alpha_p + i_{v_1}\alpha_{p+1}\right) \wedge \zeta. \tag{47}$$

Clearly, this agrees with the contraction formula defined earlier in equation (18).

We can use this contraction formula in conjunction with Cartan's magic formula to calculate the Lie derivative of a generalized form with respect to a generalized vector field. Let us first calculate the Lie derivative of a generalized form with respect to an ordinary vector field v_1. Using Cartan's formula we can write

$$L_{v_1}\overset{p}{\mathfrak{a}} = L_{v_1}\alpha_p + \left(L_{v_1}\alpha_{p+1}\right) \wedge \zeta. \tag{48}$$

We can now calculate the Lie derivative of $v_0\bar{X}$ with respect to an ordinary vector field w_1. Using the results obtained so far, we get, for any generalized p-form $\overset{p}{\mathfrak{a}}$,

$$\begin{aligned} L_{w_1}i_{v_0\bar{X}}\overset{p}{\mathfrak{a}} - i_{v_0\bar{X}}L_{w_1}\overset{p}{\mathfrak{a}} &= p(-1)^{p+1}\left(L_{w_1}v_0\right)\alpha_p \wedge \zeta \\ &= i_{(L_{w_1}v_0)\bar{X}}\overset{p}{\mathfrak{a}}. \end{aligned} \tag{49}$$

Therefore,

$$L_{w_1}(v_0\bar{X}) = \left(L_{w_1}v_0\right)\bar{X}, \qquad L_{w_1}\bar{X} = 0. \tag{50}$$

The Lie derivative of a generalized form with respect to the special vector field $\bar{X}$ can be calculated using Cartan's formula:

$$\mathcal{L}_{v_0\bar{X}}\overset{p}{\mathfrak{a}} = -pkv_0\alpha_p + (-(p+1)kv_0\alpha_{p+1} + p(-1)^{p-1}(\mathrm{d}v_0)\alpha_p + (-1)^p v_0\,\mathrm{d}\alpha_p) \wedge \zeta. \tag{51}$$

Finally, combining the results obtained so far, we get the Lie derivative of a generalized form $\overset{p}{\mathfrak{a}} = \alpha_p + \alpha_{p+1} \wedge \zeta$ with respect to a generalized vector $V = v_1 + v_0\bar{X}$:

$$\begin{aligned}\mathcal{L}_V \overset{p}{\mathfrak{a}} &\equiv (\mathbf{d} i_V + i_V \mathbf{d})\overset{p}{\mathfrak{a}} \\ &= L_{v_1}\alpha_p - pkv_0\alpha_p + \big(L_{v_1}\alpha_{p+1} - (p+1)kv_0\alpha_{p+1} \\ &\quad + p(-1)^{p-1}(\mathrm{d}v_0)\alpha_p + (-1)^p v_0\,\mathrm{d}\alpha_p\big) \wedge \zeta. \end{aligned} \tag{52}$$

Let us now try to define the Lie derivative of an ordinary vector field w_1 with respect to $v_0\bar{X}$ can be defined. It is straightforward to calculate that

$$\mathcal{L}_{v_0\bar{X}} i_{w_1}\overset{p}{\mathfrak{a}} - i_{w_1}\mathcal{L}_{v_0\bar{X}}\overset{p}{\mathfrak{a}} = i_{kv_0w_1 - (L_{w_1}v_0)\bar{X}}\overset{p}{\mathfrak{a}} - (-1)^p\big(L_{v_0w_1}\alpha_p\big) \wedge \zeta. \tag{53}$$

Because of the last term, the Lie derivative of an ordinary vector with respect to a generalized vector cannot be defined. However, we can correct the situation by modifying the Lie derivative, following a procedure similar to that used for calculating the correction term in section 5. Then we find that the Lie derivative with respect to the special vector field $v_0\bar{X}$ needs to be defined as

$$\begin{aligned}\hat{\mathcal{L}}_{v_0\bar{X}}\overset{p}{\mathfrak{a}} &= \mathcal{L}_{v_0\bar{X}}\overset{p}{\mathfrak{a}} + (-1)^p(-v_0\,\mathrm{d}\alpha_p + p(\mathrm{d}v_0)\alpha_p) \wedge \zeta \\ &= -pkv_0\alpha_p - (p+1)kv_0\alpha_{p+1} \wedge \zeta. \end{aligned} \tag{54}$$

It follows from this definition that

$$\hat{\mathcal{L}}_{v_0\bar{X}} i_{w_1}\overset{p}{\mathfrak{a}} - i_{w_1}\hat{\mathcal{L}}_{v_0\bar{X}}\overset{p}{\mathfrak{a}} = i_{kv_0w_1}\overset{p}{\mathfrak{a}}, \tag{55}$$

from which we can write

$$\hat{\mathcal{L}}_{v_0\bar{X}} w_1 = kv_0w_1. \tag{56}$$

Also, we can calculate from this definition that

$$\hat{\mathcal{L}}_{v_0\bar{X}} i_{w_0\bar{X}}\overset{p}{\mathfrak{a}} - i_{w_0\bar{X}}\hat{\mathcal{L}}_{v_0\bar{X}}\overset{p}{\mathfrak{a}} = 0, \tag{57}$$

$$\Rightarrow \hat{\mathcal{L}}_{v_0\bar{X}} w_0\bar{X} = 0. \tag{58}$$

So the complete expression for the modified Lie derivative, acting on generalized forms, is

$$\hat{\mathcal{L}}_V \overset{p}{\mathfrak{a}} = \big(L_{v_1}\alpha_p - pkv_0\alpha_p, L_{v_1}\alpha_{p+1} - (p+1)kv_0\alpha_{p+1}\big). \tag{59}$$

This Lie derivative also satisfies the Leibniz rule on wedge products of generalized forms.

Now we can directly calculate the generalized Lie derivative of a generalized vector field as

$$\hat{\mathcal{L}}_{v_1+v_0\bar{X}}(w_1 + w_0\bar{X}) = [v_1, w_1] + kv_0w_1 + L_{v_1}w_0\bar{X}. \tag{60}$$

We can also calculate the commutator of Lie derivatives to find that

$$\hat{\mathcal{L}}_{(v_1+v_0\bar{X})}\hat{\mathcal{L}}_{(w_1+w_0\bar{X})} - \hat{\mathcal{L}}_{(w_1+w_0\bar{X})}\hat{\mathcal{L}}_{(v_1+v_0\bar{X})} = \hat{\mathcal{L}}_{([v_1,w_1]+\bar{X}(L_{v_1}w_0 - L_{w_1}v_0))}. \tag{61}$$

So the generalized commutator can be defined as

$$[V, W] = [v_1, w_1] + \bar{X}\big(L_{v_1}w_0 - L_{w_1}v_0\big), \tag{62}$$

where $V = v_1 + v_0\bar{X}$, $W = w_1 + w_0\bar{X}$.

This generalized commutator satisfies Jacobi identity, as can be checked easily. Hence, we obtain all the earlier relations of sections 3, 4 and 5 using the alternative definition of generalized vector fields.

7. Application: relativistic free particle

Carlo Rovelli analysed the Hamiltonian formalism of the relativistic mechanics of a particle [8]. This analysis can be formulated in the language of generalized vector fields and generalized forms.

Let a system be described by Hamiltonian $H_0(t, q^i, p_i)$, where q^i are the coordinates, p_i are the conjugate momenta and t is the time. Now we can define an extended coordinate system as $q^a = (t, q^i)$ and corresponding momenta as $p_a = (\Pi, p_i)$. We define a curve in this extended space as $m : \tau \to (q^a(\tau), p_i(\tau))$. Here the constrained Hamiltonian will be

$$H(q^a, p_a) = \Pi + H_0(t, q^i, p_i) = 0. \tag{63}$$

Now if we consider the Hamiltonian equation for this constrained Hamiltonian, following Rovelli we can write

$$I_V \Omega = 0, \tag{64}$$

where V is a generalized vector, $V = (v_1, v_0)$, and Ω is a generalized 2-form. Let us also define Ω as

$$\Omega = (\omega, \omega\theta). \tag{65}$$

Here $\theta = p_a\, \mathrm{d}q^a$ is the canonical one-form and $\omega = \mathrm{d}\theta$. So we get

$$i_{v_1}\omega = 0, \tag{66}$$

$$-2v_0\omega + i_{v_1}(\omega\theta) = 0. \tag{67}$$

We can identify equation (66) as the same as the one considered by Rovelli and so v_1 is the tangent vector of the curve m:

$$v_1 = \dot{q}^a \frac{\partial}{\partial q^a} + \dot{p}_i \frac{\partial}{\partial p_i}, \tag{68}$$

where a dot denotes the differentiation with respect to τ. Now using equations (66), (67) and (68), we find

$$\begin{aligned} v_0 &= 1/2(p_i\dot{q}^i - H_0\dot{t}) \\ &= 1/2(p_i\dot{q}^i + \Pi\dot{t}). \end{aligned} \tag{69}$$

The function within the brackets on the right-hand side is exactly the constrained Lagrangian. So v_0 is proportional to the constrained Lagrangian of the system.

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